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Infinite-dimensional representations of the rotation group and Dirac's monopole problem

Alexander I. Nesterov and F. Aceves de la Cruz

E-mail: nesterov@cencar.udg.mx, fermin@udgphys.intranets.com

Departamento de Física, CUCEI, Universidad de Guadalajara, Av. Revolución 1500, Guadalajara, CP 44420, Jalisco, México

Abstract: Within the context of infinite-dimensional representations of the rotation group the Dirac monopole problem is studied in details. Irreducible infinite-dimensional representations, being realized in the indefinite metric Hilbert space, are given by linear unbounded operators in infinite-dimensional topological spaces, supplied with a weak topology and associated weak convergence. We argue that an arbitrary magnetic charge is allowed, and the Dirac quantization condition can be replaced by a generalized quantization rule yielding a new quantum number, the so-called *topological spin*, which is related to the weight of the Dirac string.

Keywords: Dirac string, monopole, nonassociativity, infinite-dimensional representations, indefinite metric Hilbert space.

Contents

1. Introduction

In 1931 Dirac [1] showed that a proper description of the quantum mechanics of a charged particle of the charge e in the field of the magnetic monopole of the charge q requires the quantization condition $2\mu = n$, $n \in \mathbb{Z}$ (we set $eq = \mu$ and $\hbar = c = 1$). But in spite of numerous efforts, Dirac monopole was not found so far.

Recently renewed interest in the Dirac monopole has been grown in connection with the 'fictitious' monopoles that are similar to the 'real' magnetic monopoles, however, appearing in the context of the Berry phase [2]. These type of magnetic monopoles emerge in the anomalous Hall effect of ferromagnetic materials, trapped Λ-type atoms, anisotropic spin systems, noncommutative quantum mechanics, etc., and *may carry an arbitrary 'magnetic' charge* [3, 4, 5, 6, 7, 8, 9, 10].

Widely accepted group theoretical, topological and geometrical arguments in behalf of Dirac quantization rule are based on employing classical fibre bundle theory or finite dimensional representations of the rotation group [11, 12, 13, 14, 15, 16, 17, 18]. For instance, a realization of the Dirac monopole as U(1) bundle over S^2 implies that there exists the division of space into overlapping regions $\{U_i\}$ with nonsingular vector potential

being defined in $\{U_i\}$ and yielding the correct monopole magnetic field in each region. On the triple overlap $U_i \cap U_j \cap U_k$ it holds

$$\exp(i(q_{ij} + q_{jk} + q_{ki})) = \exp(i4\pi\mu) \tag{1.1}$$

where q_{ij} are the transition functions, and the consistency condition, which is equivalent to the associativity of the group multiplication, requires $q_{ij} + q_{jk} + q_{ki} = 0 \mod 2\pi\mathbb{Z}$. This yields the Dirac selectional rule $2\mu \in \mathbb{Z}$ as a necessary condition to have a consistent U(1)-bundle over S^2 [11, 12, 14]. Thus to avoid the Dirac restrictions on the magnetic charge one needs to consider a *nonassociative* generalization of U(1) bundle over S^2. Recently we have developed a consistent pointlike monopole theory with an arbitrary magnetic charge in the framework of nonassociative fibre bundle theory [19, 20, 21, 22].

As is known, in the presence of the magnetic monopole the operator of the total angular momentum **J**, which includes contribution of the electromagnetic field, obeys the standard commutation relations of the Lie algebra of the rotation group

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$

These commutation relations fail on the Dirac string restricting the domain of definition of the operator **J** and limiting it to the functions that vanish sufficiently rapidly on the string [23, 24]. In fact, any approach to the charge quantization uses some additional assumptions, and in group theoretical treatment this is the *requirement that J_i's generate a finite-dimensional representation of the rotation group yielding* $2\mu \in \mathbb{Z}$ [25, 26, 27, 23, 28, 29]. Thus, one should give up finite-dimensional representations of the rotation group to allow an arbitrary magnetic charge.

Here we study in details the Dirac monopole problem within the framework of infinite-dimensional representations of the rotation group. The paper is organized as follows. In Section II the indefinite metric Hilbert space is introduced. In Section III the properties of infinite-dimensional representations of the rotation group are discussed. In Section IV it is argued that extending the representations of the rotation group to infinite-dimensional representations allows an arbitrary magnetic charge. In Section V the obtained results and open problems are discussed.

2. Indefinite metric Hilbert space

Starting from the early 1940s indefinite metric in the Hilbert space has been discussed and used by many authors. Recently a growing interest to this topic has been risen in the context of the so-called PT-symmetric quantum mechanics related to some non-Hermitian Hamiltonians with a real spectrum and pseudo-Hermitian operators [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40].

In conventional quantum mechanics the norm of quantum states given by

$$\int \bar{\psi}\psi dx > 0 \tag{2.1}$$

where $\bar{\psi}$ is the conjugate complex of ψ, carries a probabilistic interpretation, and appearance of an indefinite metric in Hilbert space is a sever obstacle. In particular, this leads to *negative probability of states*, that means observables with only positive eigenvalues can get negative expectation values [41].

We treat here more general situation when the normalization given by

$$\int \bar{\psi}\psi d\mu(x), \tag{2.2}$$

$d\mu$ being a suitable measure, is not necessary positive. We assume that the integral

$$\int \bar{\psi}\psi' d\mu(x), \tag{2.3}$$

may be divergent and its value is given by a regularization (for the definition of regularization of an integral see, *e.g.* [42]). There exist several possibilities of regularizing divergent integral, further we consider the regularization of the integral by analytical continuation in parameter (Sect. III), and the regularized integral will be denoted by $⨍$.

Following the notations introduced by Pauli [41], we consider an *inner product* in the indefinite metric Hilbert space $\mathcal{H}^\eta$ defined by the bilinear form of the type

$$(\psi, \psi')_\eta = (\psi, \eta\psi') = ⨍ \bar{\psi}\eta\psi' d\mu(x), \tag{2.4}$$

in which the operator η is only restricted by the condition that it has to be Hermitian and

$$⨍ \bar{\psi}\eta\psi d\mu(x) > 0. \tag{2.5}$$

The difference between our construction of the indefinite metric Hilbert space and one been suggested in [41] arises from the restriction (2.5). While Pauli requires the positive defined norm (2.2), we don't.

Let functions $\psi_m(x)$ form the basis such that

$$⨍ \overline{\psi}_m(x)\psi_{m'}(x) d\mu(x) = \eta_{mm'}$$

where $\eta_{mm'} = (-1)^{\sigma(m)}\delta_{mm'}$ is an indefinite diagonal metric, and $(-1)^{\sigma(m)} = \pm 1$ depending whether $\sigma(m)$ is even or odd. Defining the action of the operator η on ψ_m as

$$\eta\psi_m = \eta_{mm'}\psi_{m'} \tag{2.6}$$

we find that the set $\{\psi_m\}$ forms the orthonormal basis with respect to the inner product given by

$$(\psi_m, \psi_p)_\eta = ⨍ \overline{\psi}_m(x)\eta_{pm'}\psi_{m'}(x) d\mu(x) = \delta_{mp}. \tag{2.7}$$

Since the set $\{\psi_m(x)\}$ forms a basis, an arbitrary function $\psi(x) \in \mathcal{H}^\eta$ can be expanded in terms of the $\psi_m(x)$:

$$\psi(x) = \sum_m c^\eta_m \psi_m, \tag{2.8}$$

where

$$c_m^\eta = (\psi_m, \psi)_\eta = \eta_{mm'} \int \overline{\psi}_{m'}(x)\psi(x)d\mu(x) \tag{2.9}$$

Let

$$\psi'(x) = \sum_m c_m'^\eta \psi_m, \tag{2.10}$$

then the inner product $(\psi, \psi')_\eta$ can be easily calculated and the result is:

$$(\psi, \psi')_\eta = \int \bar{\psi}(x)\eta\psi'(x)d\mu(x) = \sum_m \overline{c}_m^\eta c_m'^\eta. \tag{2.11}$$

In particular, one has

$$(\psi, \psi)_\eta = \int \bar{\psi}(x)\eta\psi(x)d\mu(x) = \sum_m |c_m^\eta|^2 > 0.$$

Thus we see that the inner product in the indefinite metric Hilbert space is positive defined scalar product. This provides the standard probabilistic interpretation of the quantum mechanics.

The inner product (2.11) may be written in another form. Let us consider the sum

$$K(x, x') = \sum_m \psi_m(x)\overline{\psi_m(x')}. \tag{2.12}$$

This yields the following relations:

$$\int \psi_m(x')K(x, x')d\mu(x') = \eta_{mm'}\psi_{m'}(x), \tag{2.13}$$

$$\eta\psi'(x) = \int K(x, x')\psi'(x')d\mu(x'), \tag{2.14}$$

and it is seen that the kernel $K(x, x')$ plays here a role similar to that of δ-function in the standard Hilbert space of quantum mechanics. Now one can rewrite the inner product (2.11) as

$$(\psi, \psi')_\eta = \int\int \bar{\psi}(x)K(x, x')\psi'(x')d\mu(x)d\mu(x'). \tag{2.15}$$

The expectation value of an observable A represented by the linear operator acting in $\mathcal{H}^\eta$ is defined by

$$\langle A\rangle_\eta = \int \bar{\psi}(x)\eta A\psi(x)d\mu(x), \tag{2.16}$$

and a generalization of the Hermitian conjugate operator, being denoted as $A_\eta^\dagger$, is given by

$$A_\eta^\dagger = \eta^{-1}A^\dagger\eta \tag{2.17}$$

where $A^\dagger$ is the Hermitian conjugate operator.

Since the observables are real, we see that the related operators have to be self-adjoint in the indefinite metric Hilbert space, that means $A_\eta^\dagger = A$. In particular, applying this

to the Hamiltonian operator H, we have $H_\eta^\dagger = H$, and assuming that the wave function satisfies the Schrödinger's equation

$$i\frac{\partial\psi}{\partial t} = H\psi,$$

we obtain the conservation of the wave function normalization:

$$\frac{d}{dt}(\psi,\psi)_\eta = i\int\bar{\psi}\eta(H_\eta^\dagger - H)\psi d\mu(x) = 0. \tag{2.18}$$

Let us perform a linear transformation

$$\psi = S\psi',$$

then in order to conserve the normalization of the wave function

$$(\psi',\psi')_\eta = (\psi,\psi)_\eta$$

one has to demand

$$\eta' = S^\dagger\eta S. \tag{2.19}$$

In a similar manner we find that the observables are invariant,

$$\langle A'\rangle_\eta = \langle A\rangle_\eta, \tag{2.20}$$

if the operators transform as follows:

$$A' = S^{-1}AS, \quad A_\eta^{\dagger'} = S^{-1}A_\eta^\dagger S. \tag{2.21}$$

Assuming that, according to (2.21), the matrix A can be transformed with a suitable S to a normal form such that

$$A\psi_n = a_n\psi_n \tag{2.22}$$

we find

$$(\psi, A\psi)_\eta = \sum_n a_n|c_n^\eta|^2. \tag{2.23}$$

This leads to the conclusion that *operator with only positive eigenvalues can not have negative expectation values.* In other words in our approach, in contrast to the others recently have been developed in the growing number of papers on the subject of PT-symmetric quantum mechanics, the negative probabilities do not appear and the standard probabilistic interpretation of the quantum mechanics is preserved.

3. Infinite-dimensional representations of the rotation group

The three dimensional rotation group is locally isomorphic to the group $SU(2)$, and as well known $SO(3) = SU(2)/\mathbb{Z}_2$. In what follows the difference between $SO(3)$ and $SU(2)$ is not essential and actually we will consider $G = SU(2)$. The Lie algebra corresponding to

the Lie group $SU(2)$ has three generators and we adopt the basis $J_\pm = J_1 \pm iJ_2,\ J_3$. The commutation relations are:

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm \tag{3.1}$$

$$[J^2, J_\pm] = 0, \quad [J^2, J_3] = 0 \tag{3.2}$$

where

$$J^2 = J_3^2 + \frac{1}{2}(J_- J_+ + J_+ J_-) \tag{3.3}$$

is the Casimir operator.

Let ψ_ν^λ be an eigenvector of the operators J_3 and J^2 such that

$$J_3 \psi_\nu^\lambda = (\nu + n)\psi_\nu^\lambda, \quad J^2 \psi_\nu^\lambda = \lambda(\lambda+1)\psi_\nu^\lambda, \tag{3.4}$$

where $n = 0, \pm 1, \pm 2, \dots$, and ν, just like λ, is a certain complex number.

There are four distinct classes of representations and each irreducible representation is characterized by an eigenvalue of Casimir operator and the spectrum of the operator J_3 [43, 44, 45, 46, 47, 48]:

- *Representations unbounded from above and below*, in this case neither $\lambda + \nu$ nor $\lambda - \nu$ can be integers.

- *Representations bounded below*, with $\lambda + \nu$ being an integer, and $\lambda - \nu$ not equal to an integer.

- *Representations bounded above,* with $\lambda - \nu$ being an integer, and $\lambda + \nu$ not equal to an integer.

- *Representations bounded from above and below,* with $\lambda - \nu$ and $\lambda + \nu$ both being integers, that yields $\lambda = k/2, \quad k \in \mathbb{Z}_+$.

The nonequivalent representations in each series of irreducible representations are denoted respectively by $D(\lambda,\nu)$, $D^+(\lambda,\nu)$, $D^-(\lambda,\nu)$ and $D(\lambda)$. The representations $D(\lambda,\nu)$, $D^+(\lambda,\nu)$ and $D^-(\lambda,\nu)$ are infinite-dimensional; $D(\lambda)$ is $(2\lambda+1)$-dimensional representation. The irreducible representations $D^\pm(\lambda,\nu)$ and $D(\lambda,\nu)$ are discussed in details in [43, 44, 45, 46, 47]. Further we restrict ourselves by real eigenvalues of J_3 and the Casimir operator J^2.

The infinite-dimensional representations, considered here, are given by linear unbounded operators in infinite-dimensional linear topological spaces, supplied with a weak topology. With such a topology a weak convergence (convergence with respect a functional) - the analog of nonuniform convergence, is associated. This means that the operators $T(g)$ as elements are continuous on the group, although indeed not uniformly continuous [45, 46, 47].

3.1 Representations unbounded from above and below

Let ψ_m be eigenstates of the operators J_3 and J^2:

$$J_3\psi_m = m\psi_m, \quad J^2\psi_m = \ell(\ell+1)\psi_m \tag{3.5}$$

where ℓ and m are real numbers. Demand that the commutation relations (3.1) and (3.2) being satisfied, yields

$$J_-\psi_m = (\ell+m)\psi_{m-1}, \tag{3.6}$$

$$J_+\psi_m = (\ell-m)\psi_{m+1},. \tag{3.7}$$

Considering the invariance of an inner product $(\psi_m, \psi_{m'})$ with respect to the infinitesimal rotations generated by J_i we obtain

$$(\psi_m, (J_+ - J_-)\psi_{m'}) + (\psi_m(J_+ - J_-), \psi_{m'}) = 0, \tag{3.8}$$

$$(\psi_m, \psi_{m'}) = 0, \quad m \neq m'. \tag{3.9}$$

In particular for $m' = m+1$ one has

$$(\psi_m, J_-\psi_{m+1}) - (\psi_m J_+, \psi_{m+1}) = 0, \tag{3.10}$$

that implies

$$(l+m+1)(\psi_m, \psi_m) - (l-m)(\psi_{m+1}, \psi_{m+1}) = 0. \tag{3.11}$$

This recursion relationship can be satisfied setting

$$(\psi_m, \psi_m) = \mathcal{N}\,\Gamma(\ell-m+1)\Gamma(\ell+m+1) \tag{3.12}$$

where Γ is the gamma function and $\mathcal{N}$ is an arbitrary constant. We assume hereafter $\mathcal{N}$ be a positive constant, that does not restrict generality of consideration.

Setting

$$\mathcal{N}_m = (\mathcal{N}\,\Gamma(\ell-m+1)\Gamma(\ell+m+1))^{-\frac{1}{2}}, \tag{3.13}$$

we obtain

$$(\psi_m, \psi_m) = |\mathcal{N}_m|^{-2}(-1)^{\sigma(m)} \tag{3.14}$$

where

$$(-1)^{\sigma(m)} = \mathrm{sgn}\big(\Gamma(\ell-m+1)\Gamma(\ell+m+1)\big), \tag{3.15}$$

and $\mathrm{sgn}(x)$ is the signum function.

It follows from Eq. (3.14) that the states $|\ell, m\rangle = \mathcal{N}_m\psi_m$ form the orthonormal basis under the inner product given by

$$\langle m, \ell|\ell, m'\rangle_\eta = |\mathcal{N}_m|^2\eta_{m'm''}(\psi_m, \psi_{m'}) = \delta_{mm'}, \tag{3.16}$$

where the indefinite metric is

$$\eta_{mm'} = \delta_{mm'}\mathrm{sgn}\big(\Gamma(\ell-m+1)\Gamma(\ell+m'+1)\big) \tag{3.17}$$

It is easy to see that the operators $J_\pm$ act on the states $|\ell, m\rangle$ as

$$J_\pm|\ell, m\rangle = (\ell \mp m)\frac{\mathcal{N}_m}{\mathcal{N}_{m\pm1}}|\ell, m \pm 1\rangle \tag{3.18}$$

From here and Eqs.(3.5 - 3.7) one obtains

$$J_+|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m+1)}\;|\ell, m+1\rangle \tag{3.19}$$
$$J_-|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m-1)}\;|\ell, m-1\rangle \tag{3.20}$$
$$J_3|\ell, m\rangle = m|\ell, m\rangle \tag{3.21}$$

One can check that J_3 is the self adjoint operator in the indefinite metric Hilbert space, $J_3 = (J_3^\dagger)_\eta$, and $J_\pm = (J_\mp^\dagger)_\eta$.

Now one can start with an arbitrary vector $|\ell, m\rangle$ and apply the operators $J_\pm$ to obtain any state $|\ell, m'\rangle$. Since the eigenvalues of J_3 can be changed only by multiples of unity, one has $m = \nu + p$, $p \in \mathbb{Z}$, where ν is an arbitrary number with the fixed value within the given irreducible representation. Thus each irreducible representation $D(\ell, \nu)$ may be characterized by the given values of two invariants ℓ and ν. In fact the representations $D(\ell, \nu)$ and $D(-\ell-1, \nu)$, yielding the same value $Q = \ell(\ell+1)$ of the Casimir operator, are equivalent and the inequivalent representations may be labeled as $D(Q, \nu)$ [48].

If there exists the number p_0 such that $\nu + p_0 = \ell$, we have $J_+|\ell, \ell\rangle = 0$ and the representation becomes bounded above. In the similar manner if for a number p_1 one has $\nu + p_1 = -\ell$, then $J_-|\ell, -\ell\rangle = 0$ and the representation reduces to the bounded below. Finally, finite-dimensional unitary representation arises when there exist possibility of finding both $J_+|\ell, \ell\rangle = 0$ and $J_-|\ell, -\ell\rangle = 0$. It is easy to see that in this case 2ℓ, $2m$ and 2ν all must be integers.

Let us consider a realization of the representation $D(\ell, \nu)$ in the space of analytical functions $\mathcal{F}^{(\ell,\nu)} = \{f(z) : z \in \mathbb{C}\}$. In this realization the generators $J_\pm$ and J_3 are of the forms

$$J_- = -z^2\partial_z + 2\ell z,\;\; J_+ = \partial_z,\;\; J_3 = -z\partial_z + \ell, \tag{3.22}$$

The functions $f_m^{(\ell,\nu)}(z) = \langle z|\ell, m\rangle$ of the canonical basis of the representation $D(\ell, \nu)$ take the form

$$\langle z|\ell, m\rangle = \mathcal{N}_m z^{\ell-m}, \tag{3.23}$$

where $\mathcal{N}_m = (\Gamma(\ell - m + 1)\Gamma(\ell + m + 1))^{-1/2}$ is the normalization constant. It is easy to see that the eigenvectors $|\ell, m\rangle$ form the orhonormal basis under the inner product defined as follows:

$$\langle m, \ell|\ell, m'\rangle_\eta = \overline{\mathcal{N}}_m \mathcal{N}_{m''}\int \bar{z}^m \eta_{m'm''} z^{m''} d\mu_\ell(z) = \delta_{mm'} \tag{3.24}$$

where $\eta_{m'm''} = (-1)^{\sigma(m')}\delta_{mm'}$ is the indefinite metric, and

$$d\mu_\ell(z) = \frac{\Gamma(2\ell+2)}{2\pi i}\frac{d\bar{z}dz}{(1+|z|^2)^{2\ell+2}} \tag{3.25}$$

is the invariant measure.

The inner product of two analytic functions $f(z)$ and $g(z)$ is given by

$$\langle f|g\rangle_\eta = \int \bar{f}\eta g d\mu_\ell(z). \tag{3.26}$$

Representation T_g acts in space $\mathcal{F}^{(\ell,\nu)}$ as follows [46, 49, 50]:

$$T_g f(z) = (\bar{\alpha} + \beta z)^{2\ell} f\left(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}\right), \tag{3.27}$$

where

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1 \tag{3.28}$$

Applying (3.27) to the state $|\ell, m\rangle$, we obtain

$$\langle z|T_g|\ell, m\rangle = \mathcal{N}_m(\bar{\alpha} + \beta z)^{\ell+m}(\alpha z - \bar{\beta})^{\ell-m}. \tag{3.29}$$

It follows from (3.29) that the representation $D(\ell,\nu)$ is given over the class of analytic functions with the singularities of the branch-point type and are of orders $\ell+\nu$ and $\ell-\nu$. Note that in order to define a representation, Mittag-Leffler uniformization of the functions $f(z) \in \mathcal{F}^{(\ell,\nu)}$ must be carried out [46].

The matrix elements $\mathcal{D}^{(\ell,\nu)}_{mm'}(g)$ of the representation $D(\ell,\nu)$ are defined as the coefficients of the expansion

$$T_g|\ell, m'\rangle = \sum \mathcal{D}^{(\ell,\nu)}_{mm'}(g)|\ell.m\rangle \tag{3.30}$$

The explicit form of the matrix elements in terms of the hypergeometric function $F(a,b,c;x)$ is as follows (see Appendix A):

$$\begin{aligned}\mathcal{D}^{(\ell,\nu)}_{mm'}(g) =& e^{-im\varphi}e^{-im'\psi}(-i)^{m-m'}z^{(m'-m)/2}(1-z)^{-(m+m')/2}C^\ell_{mm'}\\ &\times F(-\ell-m, \ell-m+1, m'-m+1; z), \text{ if } m' > m \end{aligned}\tag{3.31}$$

$$\begin{aligned}\mathcal{D}^{(\ell,\nu)}_{mm'}(g) =& e^{-im\varphi}e^{-im'\psi}(-i)^{m'-m}z^{(m-m')/2}(1-z)^{-(m+m')/2}C^\ell_{m'm}\\ &\times F(-\ell-m', \ell-m'+1, m-m'+1; z), \text{ if } m' < m \end{aligned}\tag{3.32}$$

where $z = \sin^2(\theta/2)$, and

$$C^\ell_{mm'} = \frac{1}{(m'-m)!}\left(\frac{\Gamma(\ell-m+1)\Gamma(\ell+m'+1)}{\Gamma(\ell+m+1)\Gamma(\ell-m'+1)}\right)^{\frac{1}{2}}. \tag{3.33}$$

Since the spectrum of the operator J_3 has the form $m = p+\nu$, $p = 0, \pm 1, \pm 2, \ldots$, the matrix elements $\mathcal{D}^{(\ell,\nu)}_{mm'}(g)$ are s-valued functions over the group if ν is a rational number, $\nu = r/s$, and are infinity-valued if ν is irrational number.

Thus, the representation $D(\ell,\nu)$ is multiple-valued infinite-dimensional representation of the rotation group. It is given in an infinite-dimensional space, in which the convergence is of the weakest type. The elements $\mathcal{D}^{(\ell,\nu)} \in D(\ell,\nu)$ of such a space are the generalized functions, which can be considered as linear functionals on some space of basic functions. Such representations are exact representations of an infinity-sheeted universal covering group $\widetilde{SU(2)}$ of the rotation group (for details see [45, 46, 47]).

3.2 Representations bounded above

This representation is characterized by the eigenvalue ℓ of the highest-weight state: $|\ell,0\rangle$ such that $J_+|\ell,0\rangle = 0$ and $J_3|\ell,0\rangle = \ell|\ell,0\rangle$. It can be obtained formally from the representation unbounded from above and below setting $m = \ell - n$, $n = 0,1,2,\ldots$. It is convenient to consider the orthonormal states $|\ell,n\rangle$ instead of $|\ell,m\rangle$. The eigenvectors $|\ell,n\rangle$ form a basis in the space of the representation $D^-(\ell,\nu)$, where the operator J_3 acts as follows:

$$J_3|\ell,n\rangle = (\ell-n)|\ell,n\rangle, \quad n = 0,1,\ldots \tag{3.34}$$

The action of the operators $\{J_\pm\}$ on the states is given by

$$\begin{aligned} J_+|\ell,n\rangle &= \sqrt{n(2\ell-n+1))}|\ell,n-1\rangle \\ J_-|\ell,n\rangle &= \sqrt{(n+1)(2\ell-n)}|\ell,n+1\rangle \end{aligned}$$

We consider a suitable realization of the representation $D^-(\ell,\nu)$ in the space of entire analytical functions $\mathcal{F}^\ell = \{f(z) : z \in \mathbb{C}\}$. In this realization the generators $J_\pm$ and J_3 act as the first order differential operators:

$$J_- = -z^2\partial_z + 2\ell z, \; J_+ = \partial_z, \; J_3 = -z\partial_z + \ell, \tag{3.35}$$

The monomials

$$\langle z|\ell,n\rangle = \mathcal{N}_n z^n, \tag{3.36}$$

where $\mathcal{N}_n = (\Gamma(n+1)\Gamma(2\ell-n+1))^{-1/2}$ is the normalization constant, form an orthogonal basis for holomorphic functions analytical in $\mathbb{C}$, and satisfy

$$\begin{aligned} (z^n, z^p) &:= \frac{\Gamma(2\ell+2)}{2\pi i}\int \frac{\bar{z}^n z^p d\bar{z}dz}{(1+|z|^2)^{2\ell+2}} \\ &= \Gamma(n+1)\Gamma(2\ell-n+1)\delta_{np}. \end{aligned} \tag{3.37}$$

For $n > 2\ell$ the value of r.h.s. is given by the analytical continuation of the gamma function [45].

It follows from Eq.(3.37) that the states $|\ell,n\rangle$ form the orhonormal basis under the indefinite metric inner product defined as follows:

$$\langle n,\ell|\ell,p\rangle_\eta = \eta_{pp'}(\langle z|\ell,n\rangle, \langle z|\ell,p'\rangle) = \delta_{np}, \tag{3.38}$$

where $\eta_{np} = (-1)^{\sigma(n)}\delta_{np}$ and

$$(-1)^{\sigma(n)} = \begin{cases} 1, \text{ if } 2\ell - n > 0 \\ (-1)^{n+1}\,\mathrm{sgn}(\sin 2\pi\ell), \text{ if } n - 2\ell > 0 \end{cases}$$

An arbitrary state of the representation is an entire function of the type

$$f(z) = \sum_{n=0}^{\infty} f_n \langle z|\ell,n\rangle. \tag{3.39}$$

The inner product of two entire functions $f(z)$ and $g(z)$ is constructed as follows:

$$\langle f|g\rangle_\eta = \frac{\Gamma(2\ell+2)}{2\pi i}\int_D \frac{\bar{f}\eta g\, d\bar{z}dz}{(1+|z|^2)^{2+2\ell}} \tag{3.40}$$

3.3 Representations bounded below

Setting $m = n - \ell$, for the representation bounded below we have

$$J_3|\ell, n\rangle = (n - \ell)|\ell, n\rangle, \quad n = 0, 1, \dots \tag{3.41}$$

The representation is characterized by the eigenvalue ℓ of the highest-weight state: $|\ell, 0\rangle$ such that $J_-|\ell, 0\rangle = 0$ and $J_3|\ell, 0\rangle = -\ell|\ell, 0\rangle$.

The action of the operators $\{J_\pm\}$ on the state $|\ell, n\rangle$ is given by

$$J_-|\ell, n\rangle = \sqrt{n(2\ell - n + 1))}|\ell, n - 1\rangle$$
$$J_+|\ell, n\rangle = \sqrt{(n + 1)(2\ell - n)}|\ell, n + 1\rangle.$$

We consider a realization of the representation $D^+(\ell, \nu)$ in the space of analytical functions $\mathcal{F}^\ell = \{f(z) : z \in \mathbb{C}\}$, such that $z^{-2\ell} f(z)$ is the meromorphic function. In this realization the generators $J_\pm$ and J_3 act as the following differential operators:

$$J_- = -z^2\partial_z + 2\ell z, \; J_+ = \partial_z, \; J_3 = -z\partial_z + \ell, \tag{3.42}$$

The monomials

$$\langle z|\ell, n\rangle = \mathcal{N}_n z^{2\ell - n}, \tag{3.43}$$

where $\mathcal{N}_n = (n!\Gamma(2\ell - n + 1))^{-1/2}$ is the same normalization constant as in (3.36), form the orthonormal basis such that

$$\langle n, \ell|\ell, p\rangle_\eta = \frac{(-1)^{\sigma(n)}\Gamma(2\ell + 2)}{2\pi i\, n!|\Gamma(2\ell - n + 1)|} \int \frac{\bar{z}^{2\ell - n} z^{2\ell - p} d\bar{z}dz}{(1 + |z|^2)^{2\ell + 2}} = \delta_{np},$$

$\eta_{np} = (-1)^{\sigma(n)}\delta_{np}$ being the indefinite metric, and

$$(-1)^{\sigma(n)} = \begin{cases} 1, \text{ if } 2\ell - n > 0, \\ (-1)^{n+1} \operatorname{sgn}(\sin 2\pi\ell), \text{ if } n - 2\ell > 0. \end{cases}$$

An arbitrary state of the representation is a function of the type

$$f(z) = \sum_{n=0}^{\infty} f_n \langle z|\ell, n\rangle. \tag{3.44}$$

The inner product of the functions $f(z)$ and $g(z)$ is constructed as above (see Eq.(3.40)):

$$\langle f|g\rangle_\eta = \frac{\Gamma(2\ell + 2)}{2\pi i} \int \frac{\bar{f}\eta g d\bar{z}dz}{(1 + |z|^2)^{2+2\ell}}. \tag{3.45}$$

4. Infinite-dimensional representations and Dirac monopole problem

As well known any choice of the vector potential $\mathbf{A}$ being compatible with a magnetic field $\mathbf{B}$ of Dirac monopole must have singularities (the so-called strings), and one can write

$$\mathbf{B} = \mathrm{rot}\mathbf{A} + \mathbf{h},$$

where $\mathbf{h}$ is the magnetic field of the string.

There is an ambiguity in the definition of the vector potential. For instance, Dirac introduced the vector potential as [1]

$$\mathbf{A_n} = q\frac{\mathbf{r}\times\mathbf{n}}{r(r-\mathbf{n}\cdot\mathbf{r})} \tag{4.1}$$

where the unit vector $\mathbf{n}$ determines the direction of string $S_{\mathbf{n}}$, passing from the origin of coordinates to ∞ , and

$$\mathbf{h_n} = 4\pi q\mathbf{n}\int_0^\infty \delta^3(\mathbf{r}-\mathbf{n}\tau)d\tau. \tag{4.2}$$

Schwinger's choice is [51]:

$$\mathbf{A}^{SW} = \frac{1}{2}\big(\mathbf{A_n}+\mathbf{A_{-n}}\big), \tag{4.3}$$

with the string being propagated from $-\infty$ to ∞. Both vector potentials yield the same magnetic monopole field, however the quantization is different. The Dirac condition is $2\mu = p$, while the Schwinger one is $\mu = p,\ p\in\mathbb{Z}$.

These two strings are members of a family $\{S^\kappa_{\mathbf{n}}\}$ of *weighted strings*, which magnetic field is given by [1]

$$\mathbf{h}^\kappa_{\mathbf{n}} = \frac{1+\kappa}{2}\mathbf{h_n} + \frac{1-\kappa}{2}\mathbf{h_{-n}} \tag{4.4}$$

where κ is the weight of a semi-infinite Dirac string. The respective vector potential reads

$$\mathbf{A}^\kappa_{\mathbf{n}} = \frac{1+\kappa}{2}\mathbf{A_n} + \frac{1-\kappa}{2}\mathbf{A_{-n}}, \tag{4.5}$$

and since $\mathbf{A}^\kappa_{-\mathbf{n}} = \mathbf{A}^{-\kappa}_{\mathbf{n}}$, we obtain the following equivalence relation: $S^\kappa_{-\mathbf{n}} \simeq S^{-\kappa}_{\mathbf{n}}$. This implies, that the string $S^\kappa_{\mathbf{n}}$ is invariant under the following discrete transformation:

$$\kappa\to-\kappa,\ q\to-q,\ \mathbf{r}\to-\mathbf{r}. \tag{4.6}$$

Note that two arbitrary strings $S^\kappa_{\mathbf{n}}$ and $S^\kappa_{\mathbf{n}'}$ are related by

$$A^{\kappa'}_{\mathbf{n}'} = A^\kappa_{\mathbf{n}} + d\chi. \tag{4.7}$$

and vice versa. Besides, an arbitrary transformation of the strings $S^\kappa_{\mathbf{n}} \to S^{\kappa'}_{\mathbf{n}'}$ can be realized as combination $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}'}$ and $S^\kappa_{\mathbf{n}} \to S^{\kappa'}_{\mathbf{n}}$, where the first transformation preserving the weight of the string is rotation, and the second one results in changing of the weight string $\kappa\to\kappa'$ without changing its orientation [21, 52].

Let denote by $\mathbf{n}' = g\mathbf{n}, g\in \mathrm{SO}(3)$, the left action of the rotation group induced by $S^\kappa_{\mathbf{n}} \to S^\kappa_{\mathbf{n}'}$. From rotational symmetry of the theory it follows this gauge transformation can be undone by rotation $\mathbf{r}\to\mathbf{r}g$ as follows [51, 53, 21]:

$$A^\kappa_{\mathbf{n}'}(\mathbf{r}) = A^\kappa_{\mathbf{n}}(\mathbf{r}') = A^\kappa_{\mathbf{n}}(\mathbf{r}) + d\alpha(\mathbf{r};g), \tag{4.8}$$

$$\alpha(\mathbf{r};g) = \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A}^\kappa_{\mathbf{n}}(\boldsymbol{\xi})\cdot d\boldsymbol{\xi}, \quad \mathbf{r}' = \mathbf{r}g \tag{4.9}$$

[1] Previously [21, 52], we have used the other definition of the string weight, namely, $\mathbf{h}^\kappa_{\mathbf{n}} = \kappa\mathbf{h_n} + (1-\kappa)\mathbf{h_{-n}}$. To compare them, one should make substitution $\kappa\to(1+\kappa)/2$.

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where the integration is performed along the geodesic $\widehat{\mathbf{r}\,\mathbf{r}'} \subset S^2$. This gauge transformation may be written also as

$$A^{\kappa}_{\mathbf{n}'}(\mathbf{r}) = A^{\kappa}_{\mathbf{n}}(\mathbf{r}) - d\Omega(\mathbf{n}, \mathbf{n}'; \mathbf{r}), \tag{4.10}$$

where $\Omega(\mathbf{n}, \mathbf{n}'; \mathbf{r})$ is the solid angle of the geodesic simplex with the vertices $(\mathbf{n}, \mathbf{n}'; \mathbf{r})$ [12, 54, 55].

Now returning to the transformation $S^{\kappa}_{\mathbf{n}} \to S^{\kappa'}_{\mathbf{n}}$ we obtain

$$\begin{aligned} A^{\kappa'}_{\mathbf{n}} &= A^{\kappa}_{\mathbf{n}} - d\chi_{\mathbf{n}}, \\ d\chi_{\mathbf{n}} &= q(\kappa' - \kappa)\frac{(\mathbf{r} \times \mathbf{n}) \cdot d\mathbf{r}}{r^2 - (\mathbf{n} \cdot \mathbf{r})^2}, \end{aligned} \tag{4.11}$$

$\chi_{\mathbf{n}}$ being polar angle in the plane orthogonal to $\mathbf{n}$. In particular, if $\kappa' = -\kappa$ we obtain the mirror string: $S^{\kappa}_{\mathbf{n}} \to S^{\kappa}_{-\mathbf{n}} \simeq S^{-\kappa}_{\mathbf{n}}$.

For a non relativistic charged particle in the field of a magnetic monopole the total angular momentum

$$\mathbf{J} = \mathbf{r} \times (\mathbf{p} - e\mathbf{A}) - \mu\frac{\mathbf{r}}{r} \tag{4.12}$$

having the same properties as a standard angular momentum, obeys the following commutation relations:

$$[H, \mathbf{J}^2] = 0, \quad [H, J_i] = 0, \quad [\mathbf{J}^2, J_i] = 0, \tag{4.13}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{4.14}$$

where H is the Hamiltonian.

These commutation relations fail on the string [23, 24]. However, it is found that H and $\mathbf{J}$ may be extended to self-adjoint operator satisfying the commutation relations of the rotation group, with H an invariant, namely

$$[H, J_i] = 0, \quad [J_i, J_j] = i\epsilon_{ijk}J_k,$$

and this is true for any value of μ. However, requiring that the J_i generate a finite dimensional representation of the rotation group and not just the Lie algebra, one obtains that μ must be quantized and only the values $2\mu = 0, \pm 1, \pm 2, \dots$ are allowed [29].

Taking into account the spherical symmetry of the system, the vector potential can be considered as living on the two-dimensional sphere of the given radius r and being taken as [11, 12]

$$\mathbf{A}_N = q\frac{1 - \cos\theta}{r \sin\theta}\,\hat{\mathbf{e}}_{\varphi}, \quad \mathbf{A}_S = -q\frac{1 + \cos\theta}{r \sin\theta}\,\hat{\mathbf{e}}_{\varphi} \tag{4.15}$$

where (r, θ, φ) are the spherical coordinates, and while $\mathbf{A}_N$ has singularity on the south pole of the sphere, $\mathbf{A}_S$ is singular on the north one. In the overlap of the neighborhoods covering the sphere S^2 the potentials $\mathbf{A}_N$ and $\mathbf{A}_S$ are related by the following gauge transformation:

$$A_S = A_N - 2qd\varphi.$$

This is the particular case of (4.11), when $\kappa = 0$ and $\kappa' = 1$.

Choosing the vector potential as $\mathbf{A}_N$ we have

$$J_\pm = e^{\pm i\varphi}\Big(\pm \frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi} - \frac{\mu\sin\theta}{1+\cos\theta}\Big) \tag{4.16}$$

$$J_3 = - i\frac{\partial}{\partial\varphi} - \mu \tag{4.17}$$

$$\begin{aligned}\mathbf{J}^2 = &- \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \\ &+ \frac{2i\mu}{1+\cos\theta}\frac{\partial}{\partial\varphi} + \mu^2\frac{1-\cos\theta}{1+\cos\theta} + \mu^2\end{aligned} \tag{4.18}$$

where $J_\pm = J_1 \pm iJ_2$ are the raising and the lowering operators for J_3.

Now starting with Schrödinger's equation

$$\hat{H}\Psi = E\Psi, \tag{4.19}$$

and introducing $\Psi = R(r)Y(\theta,\varphi)$, we get for the angular part the following equation:

$$\mathbf{J}^2 Y(\theta,\varphi) = \ell(\ell+1)Y(\theta,\varphi). \tag{4.20}$$

By substituting

$$Y = e^{i(m+\mu)\varphi} z^{(m+\mu)/2}(1-z)^{(m-\mu)/2}F(z),$$

into Eq.(4.20), where $z = (1-\cos\theta)/2$ and m is an eigenvalue of J_3, we find that $F(z)$ is a solution of the hypergeometric equation:

$$z(1-z)\frac{d^2F}{dz^2} + \big(c - (a+b+1)z\big)\frac{dF}{dz} - abF = 0 \tag{4.21}$$

$$a = m - \ell,\ b = m + \ell + 1,\ c = 1 + m + \mu, \tag{4.22}$$

4.1 Representation bounded above or below

As is known the hypergeometric function $F(a,b;c;z)$ reduces to a polynomial of degree n in z when a or b is equal to $-n$, $(n = 0,1,2,\dots)$, and the respective solution of Eq.(4.21) is of the form [56, 50]

$$F = z^\rho(1-z)^\sigma p_n(z) \tag{4.23}$$

where $p_n(z)$ is a polynomial in z of degree n. Here we are looking for the solutions, like this of the Schrödinger equation (4.20).

The requirement of the wave function being single valued force us to take $\alpha = m + \mu$ as an integer and general solution is given by

$$Y_\ell^{(\mu,n)} = e^{i\alpha\varphi} Y_n^{(\delta,\gamma)}(u), \tag{4.24}$$

where $u = \cos\theta$, and

$$Y_n^{(\delta,\gamma)}(u) = C_n\,(1-u)^{\delta/2}(1+u)^{\gamma/2}P_n^{(\delta,\gamma)}(u), \tag{4.25}$$

$P_n^{(\delta,\gamma)}(u)$ being the Jacobi polynomials, and the normalization constant C_n is given by

$$C_n = \left(\frac{2\pi \, 2^{\delta+\gamma+1} \Gamma(n+\delta+1)\Gamma(n+\gamma+1)}{(2n+\delta+\gamma+1)\Gamma(n+1)\Gamma(n+\delta+\gamma+1)} \right)^{-1/2}$$

There are four distinct classes of solutions and each solution is characterized by an eigenvalue of Casimir operator and the eigenvalue spectrum of J_3:

$$\overset{-\,(\mu,n)}{Y}_{\ell\pm\mu} = e^{i\alpha\varphi} Y_n^{(\alpha,\beta)} \begin{cases} m = \ell - n, \ \ell + \mu \in \mathbb{Z}_+ \\ m = -\ell - n - 1, \ \ell - \mu \in \mathbb{Z}_+ \end{cases}$$

$$\overset{+\,(\mu,n)}{Y}_{\ell\mp\mu} = e^{i\alpha\varphi} Y_n^{(-\alpha,-\beta)} \begin{cases} m = n - \ell, \ \ell - \mu \in \mathbb{Z}_+ \\ m = \ell + n + 1, \ \ell + \mu \in \mathbb{Z}_+ \end{cases}$$

where we set $\alpha = m + \mu$ and $\beta = m - \mu$.

The obtained solutions belong to the indefinite metric Hilbert space $\mathcal{H}^\eta$, with the metric being

$$\eta_{np} = \begin{cases} \delta_{np}, \ \text{if } 2\ell - n > 0 \\ \delta_{np}(-1)^{n+1} \operatorname{sgn}(\sin 2\pi\ell), \ \text{if } n - 2\ell > 0. \end{cases}$$

The functions $\overset{-\,(\mu,n)}{Y}_{\ell\pm\mu}$ form the basis of the infinite-dimensional representations bounded above, being denoted respectively by $D^-(\ell,-\mu)$ and $\widetilde{D}^-(\ell,-\mu)$, and the functions $\overset{+\,(\mu,n)}{Y}_{\ell\mp\mu}$ form the basis of the representations $D^+(\ell,-\mu)$ and $\widetilde{D}^+(\ell,-\mu)$ bounded below.

A similar consideration can be done for the vector potential $\mathbf{A}_S$. In this case $\beta = m - \mu \in \mathbb{Z}$ and the corresponding wave functions

$$\overset{-(-\mu,n)}{Y}_{\ell\mp\mu} = e^{i\beta\varphi} Y_n^{(\alpha,\beta)} \begin{cases} m = \ell - n, \ \ell - \mu \in \mathbb{Z}_+ \\ m = -\ell - n - 1, \ \ell + \mu \in \mathbb{Z}_+ \end{cases}$$

$$\overset{+(-\mu,n)}{Y}_{\ell\pm\mu} = e^{i\beta\varphi} Y_n^{(-\alpha,-\beta)} \begin{cases} m = n - \ell, \ \ell + \mu \in \mathbb{Z}_+ \\ m = \ell + n + 1, \ \ell - \mu \in \mathbb{Z}_+ \end{cases}$$

form a complete set of orthonormal basis of the infinite-dimensional representation $D^\pm(\ell,\mu)$ and $\widetilde{D}^\pm(\ell,\mu)$.

Thus, we find the following series of representations:

$$\ell - \mu \in \mathbb{Z}_+ \Rightarrow \begin{cases} D^+(\ell,-\mu): \ m = n - \ell \\ D^-(\ell,\mu): \ m = \ell - n \\ \widetilde{D}^+(\ell,\mu): \ m = n + \ell + 1 \\ \widetilde{D}^-(\ell,-\mu): \ m = -\ell - n - 1 \end{cases}$$

$$\ell + \mu \in \mathbb{Z}_+ \Rightarrow \begin{cases} D^+(\ell,\mu): \ m = n - \ell \\ D^-(\ell,-\mu): \ m = \ell - n \\ \widetilde{D}^+(\ell,-\mu): \ m = n + \ell + 1 \\ \widetilde{D}^-(\ell,+\mu): \ m = -\ell - n - 1 \end{cases}$$

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where $n = 0, 1, 2, \ldots$. Notice, that while the representations $D^{\pm}(\ell, \pm\mu)$ and $D^{\pm}(\ell, \mp\mu)$ are irreducible, the representations $\widetilde{D}^{\pm}(\ell, \pm\mu)$ and $\widetilde{D}^{\pm}(\ell, \mp\mu)$ are partially reducible [48].

Taking into account the following restriction: $\ell(\ell+1) - \mu^2 \geq 0$, emerging from the Schrödinger equation, the allowed values of ℓ are found to be

$$\begin{aligned}
&\ell + \mu \in \mathbb{Z}_+ \ \Rightarrow \ \ell = -\mu + [2\mu] + k, \quad k = 0, 1, 2, \ldots \\
&\ell - \mu \in \mathbb{Z}_+ \ \Rightarrow \ \ell = \mu + k, \quad k = 0, 1, 2, \ldots
\end{aligned}$$

where $[2\mu]$ denotes the integer part of 2μ. Thus, for μ being arbitrary valued, the representations corresponding to $\ell + \mu \in \mathbb{Z}_+$ and $\ell - \mu \in \mathbb{Z}_+$ are not equivalent. However, if $2\mu \in \mathbb{Z}$, then $\ell + \mu$ and $\ell - \mu$ both are integers, and the representations has been obtained above become finite-dimensional equivalent representations, and vice versa. Thus, we see that the Dirac quantization rule is related to the finite-dimensional representations of the rotation group.

4.2 Representation unbounded from above and below

Let us start with a generic case of the weighted string $S^{\kappa}_{\mathbf{n}}$ crossing the sphere at the north and south poles. We assume $\mathbf{n} = (0, 0, 1)$. With this choice the vector potential takes the form:

$$\mathbf{A}^{\kappa} = \frac{1+\kappa}{2}\mathbf{A}_S + \frac{1-\kappa}{2}\mathbf{A}_N \tag{4.26}$$

The corresponding solution of Eq. (4.20) is given by (see Appendix B)

$$Y \propto e^{-i\kappa\mu\varphi} e^{im\varphi} z^{(m+\mu)/2} (1-z)^{(m-\mu)/2} Y_{\kappa}(z), \tag{4.27}$$

where

$$\begin{aligned}
Y_{\kappa}(z) =& \frac{1+\kappa}{2} F(m-\ell, m+\ell+1, 1+m-\mu; 1-z) \\
&+ \frac{1-\kappa}{2} F(m-\ell, m+\ell+1, 1+m+\mu; z)
\end{aligned} \tag{4.28}$$

Here m is an eigenvalue of the operator J_3, and its spectrum is of the form:

$$m = n + \nu, \ n = 0, \pm 1, \pm 2, \ldots, \tag{4.29}$$

ν being an arbitrary real number.

The set of wave functions $\left\{ Y^{(\mu,m)}_{\kappa,\ell} \right\}$ such that

$$\begin{aligned}
&Y^{(\mu,m)}_{\kappa,\ell} = C^{\ell}_{\mu m} e^{-i\kappa\mu\varphi} e^{im\varphi} z^{(m+\mu)/2} (1-z)^{(m-\mu)/2} Y_{\kappa}(z), \\
&m = n + \nu, \ n = 0, \pm 1, \pm 2, \ldots,
\end{aligned} \tag{4.30}$$

$C^{\ell}_{\mu m}$ being a normalization constant, form the complete orthonormal canonical basis of the representation $D^{\kappa}(\ell, \nu)$ in the indefinite-metric Hilbert space $\mathcal{H}^{\eta}$. The indefinite metric is given by

$$\eta_{mm'} = (-1)^{\sigma(m)} \delta_{mm'}, \tag{4.31}$$

where $(-1)^{\sigma(m)} = \mathrm{sgn}\big(\Gamma(\ell - m + 1)\Gamma(\ell + m + 1)\big)$, and there is no any restriction on μ.

Since the set of functions $\{Y^{(\mu,m)}_{\kappa,\ell}\}$ forms the orthonormal basis in the indefinite metric Hilbert space $\mathcal{H}^\eta$ of the irreducible infinite-dimensional representation $D^\kappa(\ell,\nu)$, any solution of the Schrödinger equation (4.20) can be expanded as

$$\Psi = \sum_{l,m} C_{\ell,m} Y^{(\mu,m)}_{\kappa,\ell} \tag{4.32}$$

where μ is an *arbitrary parameter*.

The obtained infinite-dimensional representation $D^\kappa(\ell,\nu)$ being in general case multivalued, depends on arbitrary parameters ν and κ, and hence, the class of all representations is too large. Further simplification can be done by relating in ν with the weight κ of the string in such a way, that in particular cases of $\kappa = \pm 1$ and $\kappa = 0$ the of functions $\{Y^{(\mu,m)}_{\kappa,\ell}\}$ being associated with the Dirac and Schwinger string, respectively, behave in appropriate way at the north and south poles of the sphere (for details see Appendix B). This leads to the following condition: $\nu = \kappa\mu$, that provides also the functions $Y^{(\mu,m)}_{\kappa,\ell}$ of the canonical basis be single-valued. Then the spectrum of the operator J_3 takes the form:

$$m = n + \kappa\mu, \; n = 0, \pm 1, \pm 2, \ldots . \tag{4.33}$$

Denoting the representation by $D(\ell,\kappa\mu)$ we find that the representation $D(\ell,\kappa\mu)$ becomes the representation $D^+(\ell,\kappa\mu)$ bounded below, if $\ell + \kappa\mu$ is an integer, and $\ell - \kappa\mu$ is not equal to an integer. In a similar way, $D(\ell,\kappa\mu)$ reduces to the representation $D^-(\ell,\kappa\mu)$ bounded above, with $\ell - \kappa\mu$ being an integer, and $\ell + \kappa\mu$ not equal to an integer.

Let us consider some particular cases related to the Dirac and Schwinger strings, starting with $\kappa = -1$. The corresponding solution can be written as follows (for details see Appendix B):

$$\begin{aligned} &Y^{(\mu,n)}_{-1,\ell} \propto e^{in\varphi} z^{n/2} (1-z)^{n/2-\mu} F(a,b,c;z), \\ &a = n - \mu - \ell, \; b = n - \mu + \ell + 1, \; c = 1 + n, \end{aligned}$$

and the function $Y^{(\mu,n)}_{-1,\ell}$ being regular at the point $z = 0$, has singularity at the south pole ($z = 1$), where the Dirac string $S^{-1}_{\mathbf{n}}$ crosses the sphere.

The choice of $\kappa = 1$ corresponds to the string $S^1_{\mathbf{n}}$ crossing the sphere at the north pole ($z = 0$), and the solution of the equation (4.20) being regular at the point $z = 1$ is given by

$$\begin{aligned} &Y^{(\mu,n)}_{1,\ell} \propto e^{in\varphi} z^{n/2+\mu} (1-z)^{n/2} F(a,b,c;1-z), \\ &a = n + \mu - \ell, \; b = n + \mu + \ell + 1, \; c = 1 + n. \end{aligned}$$

Finally, setting $\kappa = 0$ we have the Schwinger case:

$$\begin{aligned} Y^{(\mu,n)}_{0,\ell} \propto e^{in\varphi} z^{(n+\mu)/2} (1-z)^{(n-\mu)/2} \big(&F(n-\ell, n+\ell+1, 1+n-\mu; 1-z) \\ &+ F(n-\ell, n+\ell+1, 1+n+\mu; z) \big), \end{aligned} \tag{4.34}$$

with the string $S^0_{\mathbf{n}}$ being propagated from $-\infty$ to ∞.

Later on the strings $S^{-1}_{\mathbf{n}}$, $S^1_{\mathbf{n}}$ and $S^0_{\mathbf{n}}$ we will call the *fundamental strings*. The related representations are $D(\ell,\mu)$, $D(\ell,-\mu)$ and $D(\ell,0)$, and for $\ell-\mu$ or $\ell+\mu$ being integer, one obtains the representations bounded below or above: $D^{\pm}(\ell,\pm\mu)$ and $D^{\pm}(\ell,\mp\mu)$. The respective solutions we will call *generalized monopole harmonics*. When $n+\alpha$, $n+\beta$ and $n+\alpha+\beta$ all are integers ≥ 0, the generalized monopole harmonics are reduced to the *monopole harmonics* introduced by Wu and Yang [12]. The imposed here restrictions on the values of n,α and β yield the finite-dimensional unitary representation of the rotation group and the Dirac quantization condition.

Note, that the representation $D(\ell,0)$ may be realized only as infinite-dimensional representation unbounded from above and below, or finite dimensional representation when $\ell\in\mathbb{Z}$. It is easy to see that in this case μ has to be an integer, that implies Schwinger's quantization of the magnetic charge.

Returning to the general case, let us consider two strings $S^{\kappa}_{\mathbf{n}}$ and $S^{\kappa'}_{\mathbf{n}}$. The corresponding vector potentials are $\mathbf{A}^{\kappa}$ and $\mathbf{A}^{\kappa'}$ (see Eq.(4.26)), and the computation yields

$$e(A^{\kappa}-A^{\kappa'})=(\kappa'-\kappa)\mu d\varphi. \tag{4.35}$$

Recalling that $\mathbf{A}^{\kappa}$ and $\mathbf{A}^{\kappa'}$ are connected by a gauge transformation:

$$ie(\mathbf{A}^{\kappa'}-\mathbf{A}^{\kappa})=e^{-i\chi}\nabla \mathrm{e}^{i\chi}, \tag{4.36}$$

we find that the gauge transformation $\exp(i\chi)$ must satisfy

$$i\mu(\kappa-\kappa')=\frac{d}{d\varphi}(\ln e^{i\chi})$$

The solution of this equation is $\chi=\mu(\kappa-\kappa')\varphi$. If p is a winding number of the map $\exp\big(i\chi(\varphi)\big):\ S^1\rightarrow U(1)$, then

$$\mu(\kappa-\kappa')=p, \quad p\in\mathbb{Z} \tag{4.37}$$

that provides $\mu(\kappa-\kappa')$ be an integer. In particular, choosing $\kappa'=\pm1$ we obtain $\kappa\mu=\pm\mu+p$, and if $\kappa'=0$ one has $\kappa\mu=p$, $p\in\mathbb{Z}$.

We say that the strings $S^{\kappa}_{\mathbf{n}}$ and $S^{\kappa'}_{\mathbf{n}}$ are *gauge equivalent*, $S^{\kappa}_{\mathbf{n}}\simeq S^{\kappa'}_{\mathbf{n}}$, if $\kappa\mu=\kappa'\mu$ mod $\mathbb{Z}$. As we can easily see, the fundamental strings induce the following classes of gauge equivalent strings:

$$\begin{aligned}
&S^{\kappa}_{\mathbf{n}}\simeq S^1_{\mathbf{n}}: \quad \kappa\mu=\mu+p\\
&S^{\kappa}_{\mathbf{n}}\simeq S^{-1}_{\mathbf{n}}\simeq S^1_{-\mathbf{n}}: \quad \kappa\mu=-\mu+p\\
&S^{\kappa}_{\mathbf{n}}\simeq S^0_{\mathbf{n}}: \quad \kappa\mu=p,\ p=0,\pm1,\pm2,\ldots.
\end{aligned}$$

The related classes of irreducible representations are $\{D(\ell,\mu)\}$, $\{D(\ell,-\mu)\}$ and $\{D(\ell,0)\}$, respectively, and the spectrum of the operator J_3 is found to be:

$$\begin{aligned}
&D(\ell,\mu):\ m=n+\mu,\ \text{if}\ (\kappa-1)\mu\in\mathbb{Z},\\
&D(\ell,-\mu):\ m=n-\mu,\ \text{if}\ (\kappa+1)\mu\in\mathbb{Z},\\
&D(\ell,0):\ m=n,\ \text{if}\ \kappa\mu\in\mathbb{Z},\ n=0,\pm1,\pm2,\ldots.
\end{aligned}$$

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5. Discussion and concluding remarks

We have argued that a consistent pointlike monopole theory with an arbitrary magnetic charge requires infinite-dimensional representations of the rotation group, which in general case are multi-valued. Note, that in Dirac theory 'quantization of magnetic charge' follows from the requirement of the wave function be single-valued. However, the requirement of single-valuedness for a wave function is not one of the fundamental principles of quantum mechanics, and having multi-valued wave functions may be allowed until it does not affect the algebra of observables.

For the single-valued infinite-dimensional representations the generalized quantization condition is emerged, and depending on the irreducible representation $D(\ell, \nu)$, has the following form: $(1 \pm \kappa)\mu \in \mathbb{Z}$ or $\kappa\mu \in \mathbb{Z}$, where κ is the weight of the Dirac string. In particular cases $\kappa = \pm 1$ and $\kappa = 0$ we obtain the Dirac and Schwinger selectional rules respectively. The obtained quantization conditions are not mandatory, but once being introduced would give rise to a new *quantum number*, $\nu = \kappa\mu$. This quantum number being related to a winding number of the map $S^1 \to U(1)$ has a topological nature, and can be considered as the *topological spin* carried by the Dirac string.

Concerning experiment. Recently, in the context of the anomalous Hall effect, the experimental results providing evidence for the magnetic monopole in the crystal-momentum space has been reported [10]. Besides, in the literature there are others experimental proposals for the experimental probe of the 'fictitious' monopoles that are appeared in the context of the Berry phase [3, 4, 5, 6, 7, 8, 9]. The experimental observation of the anomalous scattering on the trapped Λ atom with induced magnetic monopole [4] would provide a direct confirmation of non-quantized Dirac monopole. The other possible experiment, where the existence of non-quantized Dirac string in the Berry phase of anisotropic spin systems would be proved, has been proposed in [3].

Other type of the possible experiments, where the Dirac string could be observed, is related to the scattering on magnetic charge. As was shown in [57, 58], the Dirac string is not observable from the standpoints of quantum-mechanical scattering processes, if $2\mu = n$. However, the situation is quite different for the scattering on the monopole with an arbitrary magnetic charge. In particular, for small scattering angles θ the scattering amplitude $f(\theta)$ behaves as $f(\theta) \sim \sin^{-2}(\theta/2)$, if the Dirac quantization rule holds, and this is agree with the classical results. In the case of an arbitrary μ we have $f(\theta) \sim \sin^{-2\mu}(\theta/2)$, and our numerical results show that this small angle approximation is good enough up to about $3\pi/4$ [59, 60]. Note, that this novel effect, being produced by a singular string, is a pure quantum gauge-invariant phenomena, and therefore the orientation of the Dirac string is not observable in the scattering experiments (see also [57]).

We close with some comments on observability of the Dirac string in the Aharonov-Bohm (AB) effect [61]. As is known the AB effect is appeared in quantum interference between two parts of a beam of charged particles, say electrons with charge e, passing by an infinite long solenoid. In spite of the fact that the magnetic field $\mathbf{B}$ outside the solenoid is equal to zero, it produces an interference effect at the point Q of the screen. A relative

phase shift $\Delta\varphi$ is given by

$$\Delta\varphi = e \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{r} = e\Phi, \tag{5.1}$$

where Φ is the total magnetic flux through the solenoid. The condition for the absence of observable AB effect is $e\Phi = 2\pi n,\ n \in \mathbb{Z}$.

What makes difference between the infinite long solenoid and Dirac's string is that the latter can be moved by a singular gauge transformation. This means that the monopole string can not be observed in AB experiment, if singular gauge transformations are allowed. Thus, the absence of the AB effect for the Dirac string has a crucial significance for a consistent magnetic monopole theory.

Let us assume that the beam passes in the upper half of the space divided by the plane $z = 0$. Then the contribution of the string $S^\kappa_{\mathbf{n}}$ to the relative phase shift of the wave function at the point Q is found to be

$$\Delta\varphi_+ = 2\pi(1+\kappa)\mu \tag{5.2}$$

and, if the beam passes in the lower half-space ($z < 0$), one has

$$\Delta\varphi_- = 2\pi(\kappa-1)\mu \tag{5.3}$$

It follows from the Eqs.(5.2) and (5.2) the absence of AB effect when $(1+\kappa)\mu$ and $(1-\kappa)\mu$ are integers. In the case of $\kappa \neq 0$, this yields immediately the following conditions: $2\mu \in \mathbb{Z}$, that is the celebrated Dirac rule, and quantization of the string weight, namely, $2\kappa\mu \in \mathbb{Z}$. If $\kappa = 0$, one obtains the Schwinger quantization condition, $\mu \in \mathbb{Z}$.

At first sight our results are in contradiction with the AB experiment. To clarify issue let us recall that the phase shift (5.1) arises as result of the parallel translation of wave function along the contour $\mathcal{C}$ surrounding the Dirac string. It is known that for the generators of translations the Jacobi identity fails and for the finite translations one has [13, 14]

$$\big(U_{\mathbf{a}}U_{\mathbf{b}}\big)U_{\mathbf{c}} = \exp(i\alpha_3(\mathbf{r};\mathbf{a},\mathbf{b},\mathbf{c}))U_{\mathbf{a}}\big(U_{\mathbf{b}}U_{\mathbf{c}}\big) \tag{5.4}$$

where α_s is the so-called *three cocycle*, and $\alpha_3 = 4\pi\mu \mod 2\pi\mathbb{Z}$, if the monopole is enclosed by the simplex with vertices $(\mathbf{r}, \mathbf{r}+\mathbf{a}, \mathbf{r}+\mathbf{a}+\mathbf{b}, \mathbf{r}+\mathbf{a}+\mathbf{b}+\mathbf{c})$ and zero otherwise [13]. For the Dirac quantization condition being satisfied $\alpha_3 = 0 \mod 2\pi\mathbb{Z}$, and (5.4) provides an associative representation of the translations, in spite of the fact that the Jacobi identity continues to fail. Thus, we see that the AB effect requires more careful analysis, if we assume existence of an arbitrary monopole charge.

The emerging difficulties in explanation of the AB effect may be removed by introducing nonassociative path-dependent wave function $\Psi(\mathbf{r};\gamma)$, that provides the absence of the AB effect for an arbitrary magnetic charge [22, 62].

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A. Matrix elements of representations

Here we perform computation of the matrix elements of representations has been discussed in the text.

Representations unbounded from above and below. The matrix elements of the representation $D(\ell,\nu)$ are defined as the coefficients of the expansion

$$T_g|\ell,m'\rangle = \sum \mathcal{D}^{(\ell,\nu)}_{mm'}(g)|\ell,m\rangle, \tag{A.1}$$

where (see Eq.(3.27))

$$\langle z|T_g|\ell,m'\rangle = \mathcal{N}_{m'}(\bar\alpha+\beta z)^{\ell+m'}(\alpha z-\bar\beta)^{\ell-m'}. \tag{A.2}$$

The matrix elements $\mathcal{D}^{(\ell,\nu)}_{mm'}(g)$ can be obtained as follows:

$$\mathcal{D}^{(\ell,\nu)}_{mm'}(g) = \frac{\mathcal{N}_{m'}}{\mathcal{N}_m}\frac{1}{2\pi i}\oint_\gamma(\bar\alpha+\beta z)^{\ell+m'}(\alpha z-\bar\beta)^{\ell-m'}z^{m-\ell-1}dz, \tag{A.3}$$

where

$$\frac{\mathcal{N}_{m'}}{\mathcal{N}_m} = \left(\frac{\Gamma(\ell-m+1)\Gamma(\ell+m+1)}{\Gamma(\ell-m'+1)\Gamma(\ell+m'+1)}\right)^{1/2}. \tag{A.4}$$

Making change of variables as follows: $z=(|\beta|^2t-1)/\alpha\beta t$, we find

$$\mathcal{D}^{(\ell,\nu)}_{mm'}(g) = -\frac{\mathcal{N}_{m'}}{\mathcal{N}_m}\cdot\frac{\beta^{m'-m}}{\alpha^{m'+m}}\frac{1}{2\pi i}\oint_C(-t)^{-\ell-m-1}(1-t)^{\ell+m'}(1-t|\beta|^2)^{m-\ell-1}dt, \tag{A.5}$$

where we assume $m'-m$ being an positive integer.

Using the integral representation of the hypergeometric function [63, 64]

$$F(a,b,c;z) = -\frac{\Gamma(1-a)\Gamma(c)}{\Gamma(c-a)}\cdot\frac{1}{2\pi i}\oint_C(-t)^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}dt, \tag{A.6}$$

we obtain

$$\mathcal{D}^{(\ell,\nu)}_{mm'}(g) = \beta^{m'-m}\alpha^{-(m+m')}C^\ell_{mm'}F(-\ell-m,\ell-m+1,m'-m+1;|\beta|^2), \qquad m'>m, \tag{A.7}$$

where

$$C^\ell_{mm'} = \frac{1}{(m'-m)!}\left(\frac{\Gamma(\ell-m+1)\Gamma(\ell+m'+1)}{\Gamma(\ell+m+1)\Gamma(\ell-m'+1)}\right)^{\frac12} \tag{A.8}$$

As known the hypergeometric series

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^\infty\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!} \tag{A.9}$$

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is not defined when $c = -p$ ($p = 0, 1, 2, \dots$), and for c being equal to $-p$, one has [56]

$$\lim_{c \to -p} \frac{1}{\Gamma(c)} F(a, b, c; z) = \frac{\Gamma(a+p+1)\,\Gamma(b+p+1)}{(p+1)!\,\Gamma(a)\,\Gamma(b)} z^{p+1} F(a+p+1, b+p+1, p+2; z) \tag{A.10}$$

Applying (A.10) to (A.7), we obtain in the case $m' < m$ the following expression for the matrix elements:

$$\mathcal{D}^{(\ell,\nu)}_{mm'}(g) = \beta^{m'-m} \alpha^{-(m+m')} C^{\ell}_{m'm} F(-\ell - m', \ell - m' + 1, m - m' + 1; |\beta|^2), \quad m' < m. \tag{A.11}$$

In terms of the usual Cayley-Klein parametrization [49, 65, 50]

$$\alpha = e^{i\varphi/2} \cos\frac{\theta}{2}\, e^{i\psi/2}, \quad \beta = i e^{i\varphi/2} \sin\frac{\theta}{2}\, e^{-i\psi/2}, \tag{A.12}$$

with Euler's angles being (φ, θ, ψ), the formulae (A.7) and (A.11) read

$$\begin{aligned} \mathcal{D}^{(\ell,\nu)}_{mm'}(g) =& e^{-im\varphi} e^{-im'\psi} (-i)^{m-m'} z^{(m'-m)/2} (1-z)^{-(m+m')/2} C^{\ell}_{mm'} \\ & \times F(-\ell - m, \ell - m + 1, 1 + m' - m; z), \text{ if } m' > m \end{aligned} \tag{A.13}$$

$$\begin{aligned} \mathcal{D}^{(\ell,\nu)}_{mm'}(g) =& e^{-im\varphi} e^{-im'\psi} (-i)^{m'-m} z^{(m-m')/2} (1-z)^{-(m+m')/2} C^{\ell}_{m'm} \\ & \times F(-\ell - m', \ell - m' + 1, 1 + m - m'; z), \text{ if } m' < m, \end{aligned} \tag{A.14}$$

where we set $z = \sin^2(\theta/2)$.

Representations bounded above. The matrix elements of the representation $D^-(\ell, \nu)$ are defined as the coefficients of the expansion

$$T_g|\ell, n'\rangle = \sum \mathcal{D}^{-(\ell)}_{nn'}(g) |\ell, n\rangle, \tag{A.15}$$

and may be obtained from (A.13) and (A.14) by substitution $m = \ell - n$. The computation yields

$$\begin{aligned} \mathcal{D}^{-(\ell)}_{nn'}(g) =& e^{i(n-\ell)\varphi} e^{i(n'-\ell)\psi} (-i)^{n-n'} z^{(n'-n)/2} (1-z)^{(n+n'-2\ell)/2} C^{-\ell}_{nn'} \\ & \times F(n' - 2\ell, n' + 1, 1 + n' - n; z), \text{ if } n' > n \end{aligned} \tag{A.16}$$

$$\begin{aligned} \mathcal{D}^{-(\ell)}_{nn'}(g) =& e^{i(n-\ell)\varphi} e^{i(n'-\ell)\psi} (-i)^{n'-n} z^{(n-n')/2} (1-z)^{(n+n'-2\ell)/2} C^{-\ell}_{n'n} \\ & \times F(n - 2\ell, n + 1, 1 + n - n'; z), \text{ if } n' < n \end{aligned} \tag{A.17}$$

where

$$C^{-\ell}_{nn'} = \frac{1}{(n'-n)!} \left(\frac{\Gamma(n'+1)\Gamma(2\ell - n + 1)}{\Gamma(2\ell - n' + 1)\Gamma(n+1)} \right)^{\frac{1}{2}} \tag{A.18}$$

Note that the matrix element $\mathcal{D}^{+(\ell)}_{nn'}(g)$ given in Eq.(A.16) may be rescat in terms of Jacobi's polynomials $P^{(\alpha,\beta)}_n(x)$ by the application of the following relations [56]

$$F(a, b, c; z) = (1-z)^{c-a-b} F(c-a, c-b, c; z) \tag{A.19}$$

$$F(-n, \alpha + \beta + n + 1, \alpha + 1; z) = \frac{\Gamma(n+1)\,\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} P^{(\alpha,\beta)}_n(1 - 2z) \tag{A.20}$$

to yield

$$\mathcal{D}^{-(\ell)}_{nn'}(g) = \left(\frac{\Gamma(n+1)\Gamma(2\ell-n+1)}{\Gamma(n'+1)\Gamma(2\ell-n'+1)}\right)^{1/2} e^{i(n-\ell)\varphi} e^{i(n'-\ell)\psi}(-i)^{n-n'} \times z^{\alpha/2}(1-z)^{\beta/2} P_n^{(\alpha,\beta)}(1-2z), \quad \text{if } n' > n \tag{A.21}$$

$$\mathcal{D}^{-(\ell)}_{nn'}(g) = \left(\frac{\Gamma(n'+1)\Gamma(2\ell-n'+1)}{\Gamma(n+1)\Gamma(2\ell-n+1)}\right)^{1/2} e^{i(n-\ell)\varphi} e^{i(n'-\ell)\psi}(-i)^{n'-n} \times z^{-\alpha/2}(1-z)^{\beta/2} P_n^{(-\alpha,\beta)}(1-2z), \quad \text{if } n' < n \tag{A.22}$$

where $\alpha = n' - n, \quad \beta = 2\ell - n - n'$.

Representations bounded below. The matrix elements of the representation $D^+(\ell,\nu)$ may be obtained from (A.13) and (A.14) by substitution $m = n - \ell$ and are written as follows:

$$\mathcal{D}^{+(\ell)}_{nn'}(g) = e^{i(\ell-n)\varphi} e^{i(\ell-n')\psi}(-i)^{n-n'} z^{(n-n')/2}(1-z)^{(2\ell-n-n')/2} C^{+\ell}_{nn'} \times F(-n, 2\ell-n+1, 1+n'-n; z), \quad \text{if } n' > n \tag{A.23}$$

$$\mathcal{D}^{+(\ell)}_{nn'}(g) = e^{i(\ell-n)\varphi} e^{i(\ell-n')\psi}(-i)^{n'-n} z^{(n'-n)/2}(1-z)^{(2\ell-n-n')/2} C^{+\ell}_{nn'} \times F(-n', 2\ell-n'+1, 1+n-n'; z), \quad \text{if } n' < n \tag{A.24}$$

where

$$C^{+\ell}_{nn'} = \frac{1}{(n'-n)!}\left(\frac{\Gamma(2\ell-n+1)\Gamma(n'+1)}{\Gamma(n+1)\Gamma(2\ell-n'+1)}\right)^{\frac{1}{2}} \tag{A.25}$$

The matrix elements $\mathcal{D}^{-(\ell)}_{nn'}(g)$ can be written also as

$$\mathcal{D}^{+(\ell)}_{nn'}(g) = \left(\frac{\Gamma(n+1)\Gamma(2\ell-n+1)}{\Gamma(n'+1)\Gamma(2\ell-n'+1)}\right)^{1/2} e^{i(\ell-n)\varphi} e^{i(\ell-n')\psi}(-i)^{n-n'} \times z^{-\alpha/2}(1-z)^{\beta/2} P_n^{(-\alpha,\beta)}(1-2z), \quad \text{for } n' > n \tag{A.26}$$

$$\mathcal{D}^{+(\ell)}_{nn'}(g) = \left(\frac{\Gamma(n'+1)\Gamma(2\ell-n'+1)}{\Gamma(n+1)\Gamma(2\ell-n+1)}\right)^{1/2} e^{i(\ell-n)\varphi} e^{i(\ell-n')\psi}(-i)^{n'-n} \times z^{\alpha/2}(1-z)^{\beta/2} P_n^{(\alpha,\beta)}(1-2z), \quad \text{for } n' < n. \tag{A.27}$$

B. Some properties of the generalized monopole harmonics

In this section some properties of generalized monopole harmonics, $\overset{\pm\,(\mu,n)}{Y_{\ell\pm\mu}}(\theta,\varphi)$, will be derived. Recall that the functions $\overset{+\,(\mu,n)}{Y_{\ell-\mu}}(\theta,\varphi)$ and $\overset{-\,(\mu,n)}{Y_{\ell+\mu}}(\theta,\varphi)$ form the basis of the infinite-dimensional representations $D^+(\ell,-\mu)$ and $D^-(\ell,-\mu)$, respectively. In terms of the functions

$$Y_n^{(\delta,\gamma)}(u) = C_n\,(1-u)^{\delta/2}(1+u)^{\gamma/2} P_n^{(\delta,\gamma)}(u), \tag{B.1}$$

where $P_n^{(\delta,\gamma)}(u)$ are the Jacobi polynomials, $u = \cos\theta$ and the normalization constant being

$$C_n = \left(\frac{2\pi\, 2^{\delta+\gamma+1}\Gamma(n+\delta+1)\Gamma(n+\gamma+1)}{(2n+\delta+\gamma+1)\Gamma(n+1)\Gamma(n+\delta+\gamma+1)} \right)^{-1/2} \tag{B.2}$$

one has

$$\overset{-\,(\mu,n)}{Y_{\ell+\mu}} = e^{i\alpha\varphi} Y_n^{(\alpha,\beta)}(u), \quad m = \ell - n,\ \ell+\mu \in \mathbb{Z}_+ \tag{B.3}$$

$$\overset{+\,(\mu,n)}{Y_{\ell-\mu}} = e^{i\alpha\varphi} Y_n^{(-\alpha,-\beta)}(u), \quad m = n - \ell,\ \ell-\mu \in \mathbb{Z}_+ \tag{B.4}$$

where we set $\alpha = m + \mu$ ($\alpha \in \mathbb{Z}$) and $\beta = m - \mu$.

It is easy to show that the generalized monopole harmonics satisfy the orthonormality conditions

$$\int \overline{\overset{\mp\,(\mu,n)}{Y_{\ell\pm\mu}}(\theta,\varphi)}\ \overset{\mp\,(\mu,n')}{Y_{\ell'\pm\mu}}(\theta,\varphi)\, d\Omega = \delta_{\ell\ell'}\eta_{nn'} \tag{B.5}$$

$$\int \overline{\overset{\pm\,(\mu,n)}{Y_{\ell\mp\mu}}(\theta,\varphi)}\ \overset{\mp\,(\mu,n')}{Y_{\ell'\pm\mu}}(\theta,\varphi)\, d\Omega = 0 \tag{B.6}$$

where $\eta_{nn'} = \operatorname{sgn}(\Gamma(2\ell - n + 1)\, \delta_{nn'}$. Using the well known relation $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ [66], one can find

$$\eta_{np} = \begin{cases} \delta_{np}, \text{ if } 2\ell - n > 0 \\ \delta_{np}(-1)^{n+1}\operatorname{sgn}(\sin 2\pi\ell), \text{ if } n - 2\ell > 0. \end{cases} \tag{B.7}$$

The orthonormality relations are completed by the completeness conditions

$$\sum_{\ell,n,p} \overline{\overset{\mp\,(\mu,n)}{Y_{\ell\pm\mu}}(\theta,\varphi)}\ \eta_{np}\ \overset{\mp\,(\mu,p)}{Y_{\ell\pm\mu}}(\theta',\varphi') = \frac{1}{\sin^2\theta}\,\delta(\theta-\theta')\delta(\varphi-\varphi') \tag{B.8}$$

Addition theorem

The addition theorem for generalized monopole harmonics can be obtained from the composition of successive transformations,

$$T(g'^{-1})T(g) = T(\tilde{g}) \tag{B.9}$$

where $\tilde{g} = g'^{-1}g$. Taking into account that $T(g'^{-1}) = T_\eta^\dagger(g')$, where $T_\eta^\dagger = \eta^{-1}T^\dagger\eta$, we find $T(g')_\eta^\dagger\, T(g) = T(\tilde{g})$. This yields the desired addition theorem as follows:

$$\sum_{n''}(\mathcal{D}_\eta^\dagger)_{nn''}(g')\mathcal{D}_{n''n'}(g) = \sum_{p,q,n''}\eta_{n''p}\,\overline{\mathcal{D}}_{pq}(g')\,\eta_{qn}^{-1}\,\mathcal{D}_{n''n'}(g) = \mathcal{D}_{nn'}(\tilde{g}) \tag{B.10}$$

In terms of Euler's angles, assuming $g = g(\varphi,\theta,\psi)$, $g' = g(\varphi',\theta',\psi')$ and $\tilde{g} = g(\tilde{\varphi},\tilde{\theta},\tilde{\psi})$, we obtain the well known results [58, 67]

$$\cos\tilde{\theta} = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi') \tag{B.11}$$

$$\tilde{\psi} = \psi + \bar{\psi}, \quad \tilde{\varphi} = \bar{\psi}' - \psi' \tag{B.12}$$

where $\bar\psi$ and $\bar\psi'$ being functions of θ, θ' and $\varphi - \varphi'$ are determined from

$$\tan\frac{\bar\psi+\bar\psi'}{2} = \frac{\cos\frac{\theta+\theta'}{2}}{\cos\frac{\theta-\theta'}{2}}\tan\frac{\varphi-\varphi'}{2} \tag{B.13}$$

$$\tan\frac{\bar\psi-\bar\psi'}{2} = -\frac{\sin\frac{\theta+\theta'}{2}}{\sin\frac{\theta-\theta'}{2}}\tan\frac{\varphi-\varphi'}{2} \tag{B.14}$$

Note that (B.10) can be rewritten as follows:

$$\overline{\mathcal{D}}_{nn'}(\tilde g) = \sum_{p,q,n''} \eta_{n''p}\,\mathcal{D}_{pq}(g')\,\eta^{-1}_{qn}\,\overline{\mathcal{D}}_{n''n'}(g) = \sum_{n''}\overline{\mathcal{D}}_{n''n'}(g)\,(\mathcal{D}_\eta)_{n''n}(g') \tag{B.15}$$

It easy to see that for the generalized monopole harmonics the following relation holds

$$\overset{\mp(\mu,n)}{Y_{\ell\pm\mu}}(\theta,\varphi) = \left(\frac{4\pi}{2\ell+1}\right)^{-1/2} \mathrm{e}^{-i\mu\psi}\,\overline{\mathcal{D}^{\mp(\ell)}_{n\,\ell\pm\mu}}(g) \tag{B.16}$$

Now applying (B.16), we obtain the following addition theorem for the generalized monopole harmonics:

$$\overset{\mp(\mu,n)}{Y_{\ell\pm\mu}}(\tilde\theta,\tilde\varphi) = \mathrm{e}^{i\mu(\tilde\psi-\psi)}\sum_{n'}\overline{\overset{\mp(\mu,n')}{Y_{\ell\pm\mu}}}(\theta,\varphi)\,(\mathcal{D}_\eta)_{n'n}(g') \tag{B.17}$$

Since ψ is not physical parameter, one can set in $\psi = \psi' = 0$, and inserting $n = \ell \pm \mu$ in (B.17), we obtain

$$\overset{\mp(\mu,\ell\pm\mu)}{Y_{\ell\pm\mu}}(\tilde\theta,\tilde\varphi) = \left(\frac{4\pi}{2\ell+1}\right)^{-1/2} \mathrm{e}^{i\mu\tilde\psi}\sum_{n',n''}\overline{\overset{\mp(\mu,n')}{Y_{\ell\pm\mu}}}(\theta,\varphi)\,\eta_{n'n''}\,\overset{\mp(\mu,n'')}{Y_{\ell\pm\mu}}(\theta',\varphi') \tag{B.18}$$

Note that consideration of the representation $D^{\pm}(\ell,\mu)$ and associated generalized harmonics $\overset{\pm(-\mu,n)}{Y_{\ell\pm\mu}}(\theta,\varphi)$ can be done in a similar way. In fact, the corresponding relations can be obtained from (B.4) - (B.18) by substitution $\mu \to -\mu$.

C. General solution of the hypergeometric equation and magnetic monopoles

Here we consider the general situation with a string of an arbitrary weight κ. For simplicity, we assume the string to be a straight line along the axes z, then

$$\mathbf{A}^{\kappa}_{\mathbf{n}} = \frac{1+\kappa}{2}\mathbf{A}_{\mathbf{n}} + \frac{1-\kappa}{2}\mathbf{A}_{-\mathbf{n}}, \tag{C.1}$$

and the generators of the rotation group are written as follows:

$$J_{\pm} = e^{\pm i\varphi}\bigg(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi} - \frac{(\mu+\gamma\cos\theta)\sin\theta}{1-\cos^2\theta}\bigg), \quad J_3 = -i\frac{\partial}{\partial\varphi} + \gamma$$
$$\mathbf{J}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{(\gamma+\mu\cos\theta)^2}{\sin^2\theta} - \frac{2i(\gamma+\mu\cos\theta)}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \mu^2 \tag{C.2}$$

where $\gamma = \kappa\mu$.

The general solution of the Schrödinger equation (4.20) has the form:

$$Y \propto e^{-i\gamma\varphi} e^{im\varphi} z^{(m+\mu)/2} (1-z)^{(m-\mu)/2} \big(\alpha_1 Y_1(z) + \alpha_2 Y_2(z)\big) \tag{C.3}$$

where α_1 and α_2 are arbitrary constants, and without loss of generality one can set $\alpha_1 + \alpha_2 = 1$. The functions $Y_1(z)$ and $Y_2(z)$ are linearly independent solutions of the hypergeometric equation

$$z(1-z)\frac{d^2 w}{dz^2} + (c - (a+b+1)z)\,\frac{dw}{dz} - abz = 0 \tag{C.4}$$

with $a = m - \ell,\ b = m + \ell + 1$ and $c = 1 + m + \mu$.

In the neighborhood of the singular points $z = 0, 1$ two linearly independent solutions of the hypergeometric equation are written respectively as [56]

$$w_{1(0)} = F(a,b,c;z) = (1-z)^{c-a-b} F(c-a, c-b, c; z) \tag{C.5}$$

$$w_{2(0)} = z^{1-c} F(a-c+1, b-c+1, 2-c; z) = z^{1-c}(1-z)^{c-a-b} F(1-a, 1-b, 2-c; z) \tag{C.6}$$

$$w_{1(1)} = F(a,b,a+b+1-c; 1-z) = z^{1-c} F(1+b-c, 1+a-c, a+b+1-c; 1-z) \tag{C.7}$$

$$\begin{aligned} w_{2(1)} &= (1-z)^{c-a-b} F(c-b, c-a, c-a-b+1; 1-z) \\ &= z^{1-c}(1-z)^{c-a-b} F(1-a, 1-b, c-a-b+1; 1-z) \end{aligned} \tag{C.8}$$

If one of the numbers $a, b, c-a, c-b$ is integer, then one of the hypergeometric series (C.5) – (C.8) terminates and the respective solution has the form $w = z^{\alpha}(1-z)^{\beta} P_n(z)$, where $P_n(z)$ is a polynomial of degree n.

In the text (Sec. IV) we consider the linear combination $\alpha_1 w_{1(0)} + \alpha_2 w_{1(1)}$. Using the transformation formula

$$\begin{aligned} F(a,b,c;z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b,a+b-c+1;1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F(c-a, c-b, c-a-b+1; 1-z), \end{aligned} \tag{C.9}$$

for instance, one can show that $\alpha_1 w_{1(0)} + \alpha_2 w_{1(1)}$ may be transformed to $\tilde{\alpha}_1 w_{1(0)} + \tilde{\alpha}_2 w_{2(0)}$. This support the claim that any pair of the solutions (C.5) - (C.8) can by used to obtain the general solution Y of the hypergeometric equation (C.4). In explicit form there is

$$Y = \alpha_1 F(a,b,c;z) + \alpha_2 F(a,b,a+b-c+1;1-z) \tag{C.10}$$

and in case of Dirac monopole problem one has

$$Y = \alpha_1 F(m-\ell, m+\ell+1, 1+m-\mu; 1-z) + \alpha_2 F(m-\ell, m+\ell+1, 1+m+\mu; z) \tag{C.11}$$

From the requirement of the invariance of the obtained solution with respect to the generalized parity transformation $\mu \to -\mu, \mathbf{r} \to -\mathbf{r}$ (see Sec. III), it follows $\alpha_1(\mu) =$

$\alpha_2(-\mu)$. Now, taking into account the condition $\alpha_1 + \alpha_2 = 1$, Eq. (C.1) and behavior of the hypergeometric function at the points $z = 0$ and $z = 1$, one can find that $\alpha_1 = (1+\kappa)/2$ and $\alpha_2 = (1-\kappa)/2$. This yields

$$Y = \frac{1+\kappa}{2} F(m-\ell, m+\ell+1, 1+m-\mu; 1-z) + \frac{1-\kappa}{2} F(m-\ell, m+\ell+1, 1+m+\mu; z) \quad \text{(C.12)}$$

This implies also, that $\alpha_1 - \alpha_2$ may be identified as the weight of the Dirac string, namely, $\kappa = \alpha_1 - \alpha_2$.

Finally we find that the general solution (C.2) may be written as follows:

$$Y \propto e^{i(\alpha_2-\alpha_1)\mu\varphi} e^{im\varphi} z^{(m+\mu)/2}(1-z)^{(m-\mu)/2}\big(\alpha_1 F(m-\ell, m+\ell+1, 1+m-\mu; 1-z) \\ + \alpha_2 F(m-\ell, m+\ell+1, 1+m+\mu; z)\big)$$

where $\alpha_1 + \alpha_2 = 1$.

Let us consider the case $\alpha_1 = 1$ and $\alpha_2 = 0$ being associated with the Dirac string passing from the origin of coordinates to ∞. The corresponding solution of the Schrödinger equation is given by

$$Y^{(\mu,m)}_{1,\ell} \propto e^{-i\mu\varphi} e^{im\varphi} z^{(m+\mu)/2}(1-z)^{(m-\mu)/2} F(m-\ell, m+\ell+1, 1+m-\mu; 1-z) \quad \text{(C.13)}$$

should be singular at the north pole ($z = 0$), where the string crosses the sphere, and regular at the south pole ($z = 1$).

The computation yields $c-a-b = -(m+\mu)$, and according to the general properties of the hypergeometric function one can see that $Y^{(\mu,m)}_{1,\ell}$ goes to infinity like $z^{-|m+\mu|/2}$, $z \to 0$. Now taking into account the well known relation

$$\lim_{c\to -p} \frac{1}{\Gamma(c)} F(a, b, c; z) = \frac{\Gamma(a+p+1)\,\Gamma(b+p+1)}{(p+1)!\,\Gamma(a)\,\Gamma(b)} z^{p+1} F(a+p+1, b+p+1, p+2; z) \quad \text{(C.14)}$$

we find that the obtained solution has no singularity at $z = 1$, if $m - \mu$ is an integer.

In a similar way, choosing $\alpha_1 = 0$ and $\alpha_2 = 1$, we find that the corresponding solution

$$Y^{(\mu,m)}_{-1,\ell} \propto e^{i\mu\varphi} e^{im\varphi} z^{(m+\mu)/2}(1-z)^{(m-\mu)/2} F(m-\ell, m+\ell+1, 1+m+\mu; z) \quad \text{(C.15)}$$

being singular at the south pole($z = 1$) is regular at $z = 0$, if $m + \mu$ is an integer. Note that in both cases the solutions are given by single-valued functions.

Finally, the choice of $\alpha_1 = \alpha_2 = 0$ being related to the Schwinger string, yields

$$Y^{(\mu,m)}_{0,\ell} \propto e^{im\varphi} z^{(m+\mu)/2}(1-z)^{(m-\mu)/2}\big(F(m-\ell, m+\ell+1, 1+m-\mu; 1-z) \\ + F(m-\ell, m+\ell+1, 1+m+\mu; z)\big). \quad \text{(C.16)}$$

By requiring $Y^{(\mu,m)}_{0,\ell}$ be a single-valued function, we obtain $m \in \mathbb{Z}$.

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